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The generation of continental shelf waves

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Assuming zero divergence, the equations of forced long waves in a uniformly rotating, homogeneous ocean are reduced to a single partial differential equation for the stream function. A shelf of exponential slope between a rigid continent and a sea of uniform depth is taken as a model, and certain other assumptions are made which appear physically reasonable. Calculations made on the basis of this simplified theory are in good qualitative agreement with observations of shelf waves, indicating that these waves are generated by the stress of the longshore component of the geostrophic wind.

1. Introduction

Certain recent investigations of the response of sea level to variations in weather conditions of periods greater than the pendulum day have been principally concerned with the relation between sea level and atmospheric pressure. A global survey of monthly mean sea levels (Pattullo, Munk, Revelle & Strong 1955) found general agreement with the 'isostatic' model, in which it is assumed that the relation between sea level, atmospheric pressure and water density is such that the total pressure at any fixed point on the bottom in sufficiently deep water is constant. Thus, for constant water density, an increase of one millibar in atmospheric pressure should produce a decrease of 1·01 cm in sea level.

Hamon (1962, 1963, 1966) has shown that, for periods of several days, the response of sea level at the Australian coastline to variations in atmospheric pressure is consistently smaller (in magnitude) on the east and larger on the west coast than that predicted by the isostatic model. Hamon has also investigated the lagged correlations of sea level between successive stations. These indicate the existence of waves of amplitudes of several centimetres travelling northward along the east coast with a velocity of 350 cm/sec and southward along the west coast with a velocity between 300 and 600 cm/sec. Mooers & Smith (1967) have found similar waves on the west coast of the United States, travelling northward with a velocity of 250 cm/sec. Making certain physical assumptions, Robinson (1964) constructed a model of continental shelf waves and suggested that the waves observed by Hamon were examples of these, the direction and velocity of shelf waves being consistent with the observations. In an extension of this theory, Mysak (1967*a*, *b*) proposes that the anomalous behaviour of sea level may be explained by the generation of shelf waves in resonance with pressure variations.

There are a number of very serious objections, however, to the argument that the waves are generated by pressure variations. In the light of these, it is necessary to look for an alternative causal mechanism. It will be shown in this paper that the longshore component of the stress of the geostrophic wind can generate sufficient vorticity to explain many of the important features of the observations. The analysis is only approximate, in that not only are the effects of friction and stratification neglected, but also some rather arbitrary assumptions are made about the geostrophic wind. Nevertheless, it is our opinion that the results in this paper give good support to the belief that the geostrophic wind is the main cause of the shelf wave phenomenon.

2. The generation of shelf waves by wind stress

The equations of motion and continuity in the linearized, shallow-water wave theory, neglecting bottom friction and internal dissipative forces, are:

$$\frac{\partial u}{\partial t}-fv = -g\frac{\partial(\zeta+\phi)}{\partial x}+\frac{\tau_x}{h}, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+fu = -g\frac{\partial(\zeta+\phi)}{\partial y}+\frac{\tau_y}{h}, \tag{2.2}$$

$$\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}+\frac{\partial\zeta}{\partial t} = 0. \tag{2.3}$$

In these equations, u and v are the depth-averaged components of velocity in the x, y directions respectively, ζ is the elevation of the sea surface above the equilibrium level, and τ_x, τ_y are the components of the stress acting on this surface due to the geostrophic wind. Also, ϕ represents the atmospheric pressure measured in cm of water, h is the depth, g the acceleration due to gravity and f is the Coriolis parameter. Equations (2.1) to (2.3) are expressed in c.g.s. units, with the density of water equal to 1 g/cm^3.

To examine the properties of continental shelf waves, consider an infinite straight coastline parallel to the y-axis and let the depth h be a function of x only, that is, all bottom contours are parallel to the coastline. In a previous paper (Buchwald & Adams 1968, which will be referred to as paper I) it was shown that for the time and length scales of this problem, it is sufficient to assume that f is constant, and that the motion is horizontally non-divergent, thereby neglecting the third term of (2.3). A more cumbersome analysis (Adams 1968) shows that the inclusion of this term has a negligible effect on the solution. It is now possible to express u, v in terms of a stream function ψ by

$$u = D\frac{\partial\psi}{\partial y}, \quad v = -D\frac{\partial\psi}{\partial x}, \tag{2.4}$$

where $D = h^{-1}$. Differentiating (2.1) with respect to y, (2.2) with respect to x, subtracting, and using (2.4), we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{D'}{D}\frac{\partial\psi}{\partial x}\right]-f\frac{D'}{D}\frac{\partial\psi}{\partial y} = \frac{\partial\tau_x}{\partial y}-\frac{\partial\tau_y}{\partial x}-\frac{D'}{D}\tau_y, \tag{2.5}$$

where the primes represent differentiation with respect to x. Note that the 'isostatic' model is included in the non-divergent approximation, with $\zeta = -\phi$, $\psi = a$ constant and $\tau_x = \tau_y = 0$.

Following paper I, we consider a shelf profile of the form

$$D(x)\begin{cases} = D_0 e^{-2bx} & (0 < x < l), \\ = D_0 e^{-2bl} & (x > l), \end{cases} \tag{2.6}$$

in the 'shelf' and 'ocean' regions respectively, so that in (2.5) the quantity D'/D has a constant value in each region. Assume harmonic variation $e^{i\omega t}$ $(\omega > 0)$ with time and take Fourier transforms with respect to y, such that

$$2\pi A(\kappa) = \int_{-\infty}^{\infty} a(y)\, e^{-i\kappa y}\, dy, \tag{2.7a}$$

which may be inverted by

$$a(y) = \int_{-\infty}^{\infty} A(\kappa)\, e^{i\kappa y}\, d\kappa. \tag{2.7b}$$

In equation (2.5), $\partial/\partial t$, $\partial/\partial y$ may now be replaced by $i\omega$, $i\kappa$ respectively and, using (2.6), we obtain the transformed equations

$$\Psi_s'' - 2b\Psi_s' + \left(\frac{2bf\kappa}{\omega} - \kappa^2\right)\Psi_s = \frac{\kappa}{\omega} T_x - \frac{1}{i\omega} T_y' + \frac{2b}{i\omega} T_y, \tag{2.8a}$$

$$\Psi_0'' - \kappa^2 \Psi_0 = \frac{\kappa}{\omega} T_x - \frac{1}{i\omega} T_y', \tag{2.8b}$$

where $\Psi(x, \kappa)$, $T_x(x, \kappa)$, $T_y(x, \kappa)$ are the Fourier transforms of ψ, τ_x, τ_y, and the subscripts s, 0, refer to the shelf and ocean regions, respectively. It is to be understood that $e^{i\omega t}$ is a factor in all the dependent variables. Following paper I, the boundary conditions are that

$$\Psi_s = 0 \quad \text{at} \quad x = 0, \tag{2.9a}$$

$$\Psi_s = \Psi_0, \quad \Psi_s' = \Psi_0' \quad \text{at} \quad x = l. \tag{2.9b}$$

For a given wind-stress it is now possible to solve the system of equations (2.8), (2.9), and hence obtain an expression for the Fourier transform of ψ. However, this expression is rather complicated and inversion of the transform proves to be quite difficult. It is, therefore, desirable at this stage to make some approximations which lead to simpler expressions. We make the following assumptions on physical grounds: (a) assume that the shelf-waves are generated by the longshore component of the wind stress in the shelf region only. In general, the horizontal dimensions of weather systems are large compared with the shelf width. Now the right-hand side of (2.8) consists of the curl of the windstress, and, in the shelf region only, a contribution to the vorticity due to the change in depth across the shelf. Because of the scales involved, the latter term dominates, and it is, therefore, assumed that we may neglect the right-hand sides of both (2.8a) and (2.8b), except for the term $2bT_y/i\omega$. The same reasoning can be used to justify an assumption that T_y is constant across the width of the shelf. It should be noted that both the above assumptions have been tested easily enough in given numerical examples, and were found to be satisfactory; (*b*) the observations of shelf waves indicate that the wavelengths are large. In paper I it was shown that for the shelf on the east coast of Australia, the theoretical lengths corresponding to periods of about 7 days are around 1500 km, and 15 km, and it is suggested that the contribution from the short waves must be comparatively small. Hence we concentrate our attention on the long wave end of the spectrum, and assume that $\Psi(x,\kappa)$ is very small for $|\kappa| \leqslant \kappa_0$, where $\kappa_0 \ll b$. This assumption implies that the term in κ^2 may be neglected on the left-hand side of (2.8*a*), and, more important, the boundary condition at $x = l$ is $\Psi' = 0$. Justification of the latter conclusion is given in paper I.

The result of making assumptions (*a*) and (*b*) is to reduce the problem to the determination of the amplitudes of the long waves generated on the shelf by the longshore component of the wind. If it can be established that the amplitudes and phases of these waves agree qualitatively with the observations, then we have obtained a reasonable theoretical explanation.

After making the approximations, (2.8) and (2.9) reduce to

$$\Psi'' - 2b\Psi' + \gamma\Psi = (2b/i\omega)\, T_y, \tag{2.10}$$

for $0 < x < l$, where T_y is independent of x,

$$\gamma = 2bf\kappa/\omega,$$

and the boundary conditions are

$$\Psi(0,\kappa) = 0, \quad \Psi'(l,\kappa) = 0. \tag{2.11}$$

The solution of (2.10) which satisfies (2.11) is

$$\Psi(x,\kappa) = A\, e^{bx} \sin \beta x + B(1 - e^{bx} \cos \beta x), \tag{2.12}$$

where

$$\beta^2 = \gamma - b^2, \tag{2.13}$$

$$B = 2bT_y/i\omega\gamma, \tag{2.14}$$

and

$$A = \frac{b\cos\beta l - \beta \sin\beta l}{b \sin\beta l + \beta\cos\beta l} B. \tag{2.15}$$

Hence

$$\psi(x,y,t) = e^{i\omega t} \int_{-\infty}^{\infty} \Psi(x,\kappa)\, e^{i\kappa y}\, d\kappa \tag{2.16}$$

can be obtained for any given T_y.

3. The line wind stress

As a first step in the solution of the problem we calculate the appropriate Green's function by assuming that

$$\tau_G(y,\eta) = \tau_y(\eta)\, \delta(y-\eta), \tag{3.1}$$

where $\delta(y-\eta)$ is the Dirac delta function, so that

$$\tau_y(y) = \int_{-\infty}^{\infty} \tau_G(y,\eta)\, d\eta,$$

and it is assumed that τ_y is independent of x in the shelf region. Taking the transform with respect to y,

$$T_G(\kappa,\eta) = \tau_y(\eta)\, e^{-i\kappa\eta}/2\pi,$$

so that, in (2.12), $$B = b\tau_y(\eta)\, e^{-i\kappa\eta}/\pi i\omega\gamma.$$

Hence $$\psi_G(x,y,t,\eta) = \frac{b}{\pi i\omega}\tau_y(\eta)\int_{-\infty}^{\infty}\left[\frac{b\cos\beta l-\beta\sin\beta l}{b\sin\beta l+\beta\cos\beta l}e^{bx}\sin\beta x \ldots + 1 - e^{bx}\cos\beta x\right]e^{i\kappa(y-\eta)}\,(d\kappa/\gamma), \quad (3.2)$$

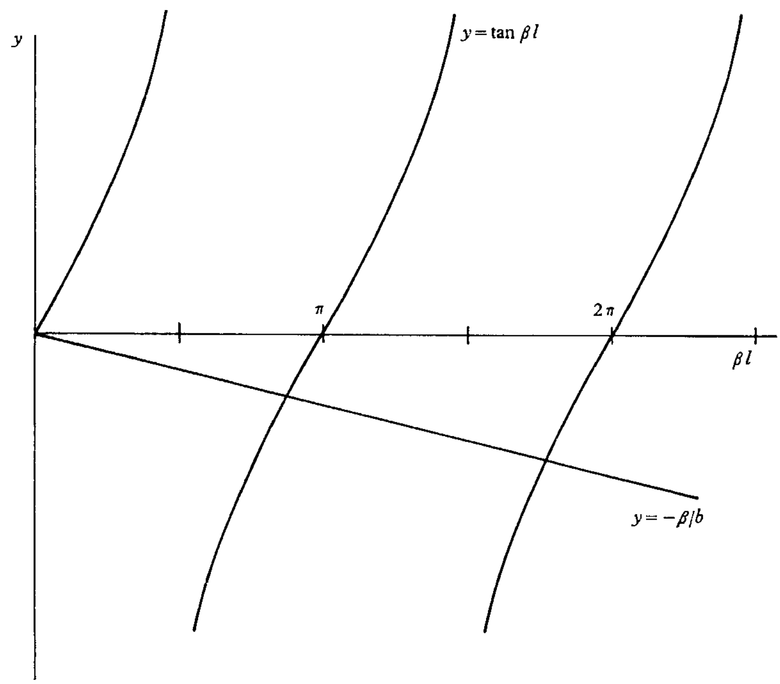

FIGURE 1. Intersections of $y = \tan\beta l$ with $y = -\beta/b$.

where β is given in (2.13) and $\tau_y(\eta)$ is understood to contain the factor $e^{i\omega t}$.

Now in (2.2), let $$p(y,t) = \zeta(0,y,t)+\phi(0,y,t), \quad (3.3)$$

so that $p(y,t)$ is the 'non-isostatic' part of the sea level observed at the coast, and let

$$P(\kappa,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(y,t)\, e^{-iy\kappa}\, dy$$

be the Fourier transform of $p(y,t)$. Take the transform of (2.2) and let $x = 0$, with the result that, after some computation,

$$P_G(\kappa,t,\eta) = \frac{D_0\tau_y(\eta)\, e^{-i\kappa\eta}}{2\pi i g\kappa}\frac{\beta\cos\beta l-b\sin\beta l}{\beta\cos\beta l+b\sin\beta l}, \quad (3.4)$$

and $$p_G(y,t,\eta) = \int_{-\infty}^{\infty} P_G(\kappa,t,\eta)\, e^{i\kappa y}\, d\kappa. \quad (3.5)$$

The integrand in (3.5) is $O(\kappa^{-1})$, as $\kappa \to \infty$, and has simple poles on the real axis at $\kappa = 0$, and at the roots of

$$\Delta(\kappa) = b \sin \beta l + \beta \cos \beta l = 0. \tag{3.6}$$

It is easily seen from figure 1 that there is a root β_n of (3.6) in each interval $(n-\frac{1}{2})\pi < \beta l < n\pi$, where $n = 1, 2, 3, \ldots$, and that $\beta_n l \to (n-\frac{1}{2})\pi$, as $n \to \infty$. Let κ_n, γ_n be the values of κ, y which correspond to $\beta = \beta_n$.

In Australian latitudes f is negative, and takes a value of about

$$f = -8{\cdot}3 \times 10^{-5}\,\text{sec}^{-1}$$

at latitude 35° S. It is seen from (2.13) that $\gamma > 0$ for all real β, so that all the κ_n lie on the negative real axis in the complex κ plane. There is, therefore, an apparent ambiguity in evaluating (3.5), and in order to ensure that only outgoing waves are obtained, it is convenient to resort to the usual device of assuming that

$$\omega = \nu - i\epsilon \quad (\epsilon > 0),$$

where $\epsilon \ll \nu$. The integral in (3.5) can then be evaluated by residues without ambiguity, and the solution for real ω is then obtained by allowing $\epsilon \to 0$. Now, from (3.6)

$$\frac{\partial \beta}{\partial \omega} d\omega + \frac{\partial \beta}{\partial \kappa} d\kappa = 0,$$

where $d\omega = -i\epsilon$. Hence the displacement of the pole at $\kappa = \kappa_n$ is given by

$$d\kappa = -i\epsilon\kappa_n/\omega,$$

so that, since $\kappa_n < 0$, all the poles on the negative real axis are displaced into the upper half plane.

The pole at $\kappa = 0$ is somewhat troublesome, in that it occurs on account of the non-divergent approximation and has no real physical meaning. If this approximation is valid then one would expect only very small contributions from this singularity. It will be shown, in fact, that this is so, but, for the sake of mathematical completeness, we shall include the contribution from this pole along with the others in the upper half plane, i.e. we assume that the path of integration in (3.5) is below the pole at $\kappa = 0$. Note also that the integrand is an even function of β, so that there is no singularity at $\beta = 0$.

The integral in (3·5) can now be evaluated by completing the path of integration by means of a large semi-circle, in the upper half plane for $y > \eta$, and in the lower half plane for $y < \eta$. The contribution of the integral along the semi-circles is vanishingly small by Jordan's lemma, so that the integrals in (3.5) can be replaced by $2\pi i$ times the sum of the residues at the poles in the appropriate half plane, with the result that

$$g p_G(y, t, \eta) = D_0 \tau_y(\eta) \left[\alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{i\kappa_n(y-\eta)} \right] H(y-\eta), \tag{3.7}$$

where $H(y-\eta)$ is the Heaviside unit function, and

$$\alpha_0 = D_1/D_0, \quad \alpha_n = 2\omega^2\beta_n^2/bf\kappa_n(2fl\kappa_n + \omega). \tag{3.8}$$

For the shelf on the east coast typical values of the parameters assumed in paper I are $l = 80$ km, $D_0 = 4{\cdot}4 \times 10^{-4}\,\text{cm}^{-1}$, $D_1 = 2 \times 10^{-6}\,\text{cm}^{-1}$, $bl = 2{\cdot}7$, whence

$\alpha_0 = 4{\cdot}5 \times 10^{-3}$, to two significant figures. Calculated values of $\beta_n l$, $\kappa_n l$, and α_n are given in table 1, to two significant figures, for the first three modes, in the case $f/\omega = 7$.

4. The wind stress model

On the average, the weather features in Australia consist of a series of anti-cyclones, travelling from west to east across the continent. Suppose we represent these by a pressure distribution of the form

$$\phi = \Phi(y)\, e^{i(ax+\omega t)}, \tag{4.1}$$

Mode	$\beta_n l$	$\kappa_n l$	α_n
1	2·4	−0·35	0·26
2	5·2	−0·9	0·24
3	8·2	−2·2	0·10

TABLE 1. Approximate values of the wave-number and coefficient α_n for the first three modes.

where, in the local co-ordinate system used in this paper, $a > 0$ on the west coast and $a < 0$ on the east coast. It follows that the *isostatic* response observed at the coast is of the form, (taking the real part),

$$\zeta_I = -\Phi(y)\cos\omega t. \tag{4.2}$$

The geostrophic part of the longshore component of the wind is given by

$$W = \lambda\frac{\partial\phi}{\partial x} = \lambda\,|a|\,\Phi(y)\,i(ax+\omega t+\tfrac{1}{2}\pi\,\mathrm{sgn}\,a), \tag{4.3}$$

so that the wind leads the pressure by a phase angle of $\frac{1}{2}\pi$ on the east coast, and trails by the same amount on the east coast. In (4.3), λ is a constant which need not be determined here.

It is beyond the scope of this paper to attempt to find a relationship between W and the corresponding wind stress, but in view of (4.3) and assumptions discussed earlier in this paper, it is reasonable to take for a model of the y-component of the wind stress

$$\tau_y = \tau(y)\, e^{i(\omega t+\frac{1}{2}\pi\delta)}, \tag{4.4}$$

where $\delta = 1$ on the west coast, $\delta = -1$ on the east coast, and $\tau(y)$ is some appropriate positive function which gives an estimate of the stress for a given geostrophic wind. Now suppose that $\tau(y)$ can be represented by the rectangular function

$$\tau(y) = \begin{cases} 0 & (y < 0), \\ \sigma & (0 < y < L), \\ 0 & (y > L). \end{cases} \tag{4.5}$$

In order to obtain an estimate of σ we use the experimental law

$$\sigma = c\rho_a U^2\,\text{dynes/cm}^2,$$

where ρ_a is the density of air (taken as $1{\cdot}25 \times 10^{-3}\,\mathrm{g\,cm^{-3}}$), U is the speed of the wind in cm/sec, and a moderate estimate for c is $c = 1{\cdot}5 \times 10^{-3}$. Thus for a supposed geostrophic wind of amplitude 7 m/sec, $\sigma = 0{\cdot}9\,\mathrm{dynes/cm^2}$, approximately, and this seems to be a reasonable estimate of the amplitude of the longshore component of the stress of the geostrophic wind.

Note that in (4.5), L is of the order of the dimensions of the weather systems, so that $L > 10^8\,\mathrm{cm}$, probably. Observations are generally made in the region $0 < y < L$, but, as will be seen later on, the region $y > L$ is also of some theoretical and experimental interest.

The actual displacement at the shore, $p(y,t)$, as defined in (3.3), is now obtained by performing the integration

$$p(y,t) = \int_{-\infty}^{\infty} p_G(y,t,\eta)\,d\eta, \tag{4.6}$$

where $p_G(y,t,\eta)$ is given by (3.7) and (4.5). It is convenient to discuss two separate cases.

Case (a): $0 < y < L$

In this case (4.6) reduces to

$$\begin{aligned} gp(y,t) &= \sigma D_0\, e^{i(\omega t+\frac{1}{2}\pi\delta)} \int_0^y \left[\alpha_0 + \sum_{n=1}^{\infty} \alpha_n\, e^{i\kappa_n(y-\eta)}\right] d\eta \\ &= \sigma D_0\, e^{i(\omega t+\frac{1}{2}\pi\delta)} \left[\alpha_0 y + i \sum_{n=1}^{\infty} \frac{\alpha_n}{\kappa_n}(1 - e^{i\kappa_n y})\right]. \end{aligned} \tag{4.7}$$

In the units used so far $L = 12l$, or thereabouts, so that $y < 12l$. Now

$$|\alpha_1/\kappa_1\alpha_0| \approx 160,$$

whence we may conclude that the effects of taking zero divergence are negligible for most values of y, and the term in α_0 can be ignored. It is also evident from table 1 that the second mode has about one-third the amplitude of the first mode, and that the third and higher modes have amplitudes which are small enough to be neglected.

Concentrating attention on the first mode, and taking the real part,

$$\begin{aligned} p_1(y,t) &= -\frac{\delta\sigma D_0\alpha_1}{\kappa_1 g}[\cos\omega t - \cos(\omega t + \kappa_1 y)] \\ &= \frac{2\delta\sigma D_0\alpha_1}{\kappa_1 g} \sin(\omega t + \tfrac{1}{2}\kappa_1 y)\sin(\tfrac{1}{2}\kappa_1 y). \end{aligned} \tag{4.8}$$

There are a number of conclusions which can be reached from (4.8), and these are listed below. (i) The maximum amplitude reached by the first mode is $|2\sigma D_0\alpha_1/\kappa_1 g|$, which, for the parameters assumed for the east coast, is 6 cm, to the nearest cm. This is of the same order of magnitude as Hamon's (1966) observations on the east coast. On the west coast $l \approx 160\,\mathrm{km}$, and, assuming similar geometry for the shelf profile, so that the value of bl remains about $2{\cdot}5$, the expected amplitude of shelf waves is roughly doubled. Again this is in agreement with observations of the barometer factor at Fremantle. (ii) Noting that $\kappa_1 < 0$ in the southern hemisphere, and $\delta = \mp 1$ on the east and west coasts respectively, we see that p_1 is positive in the east and negative in the west. This is in accord with the observations. In other words, the difference in phase of the observations of shelf waves is due to the fact that anticyclonic weather systems approach the west coast from the ocean, and the east coast from the land, so that a 'positive' geostrophic wind precedes the pressure in the east, but lags in the west. (iii) The maximum amplitude is reached when $y = \pi/\kappa_1$, that is, at a distance of about 700 km from $y = 0$, and an amplitude of about 3 cm is reached at a distance of about 250 km. (iv) It can be shown that calculations of velocity from observations by the phase lag method yield results which depend on the wind-stress profile. Suppose observations are made at two stations with co-ordinates $y, y+Y$, then, in principle, the method is to compute, from the observations, the stationary values of

$$\phi(y, Y, T) = \int_{t_0}^{t_0+\Delta} p_1(y,t)\, p_1(y+Y, t+T)\, dt$$

with respect to the variable T. The observations are made over many periods, so that $\Delta \gg 2\pi/\omega$. If we now substitute the second of (4.8) into the above expression, and carry out the integration with respect to t, we have

$$\phi = \frac{2\delta\sigma D_0 \alpha_1}{\kappa_1 g} \sin(\tfrac{1}{2}\kappa_1 y) \sin\{\tfrac{1}{2}\kappa_1(y+Y)\}$$

$$\times \left\{\frac{1}{2\omega} \sin\Big[(2\omega t + \mu)\Big]_{t_0}^{t_0+\Delta} - \Delta \cos(\omega T + \tfrac{1}{2}\kappa_1 Y)\right\},$$

where $\mu = \omega T + \kappa_1(y + \frac{1}{2}Y)$. Since $\Delta \gg 2\pi/\omega$, $|\phi|$ has a maximum when $\omega T + \frac{1}{2}\kappa_1 Y = 0$, i.e. when $Y/T = -2\omega/\kappa_1$. Thus, for the stress profile assumed in this paper, an analysis of the observations by the phase lag method would give a phase velocity of $-2\omega/\kappa_1 \approx 540$ cm/sec, although the wave velocity of free shelf waves is about 270 cm/sec. The reason for the apparent discrepancy is that there is interference between the generated and propagating waves in the region where the waves are being actually generated. The phase lags calculated by Hamon (1966) for the east coast indicate that the observed velocity of shelf waves is about 350–400 cm/sec.

The theoretical phase velocity is very dependent on the form of the function $\tau_y(y)$, and the step function chosen in this paper is not very realistic physically. It is, however, possible to reach the very important conclusion that not too much significance can be attached to observations of the velocity of shelf waves, at least not until the shape of the function $\tau_y(y)$ is known with reasonable accuracy.

Case (b): $y > L$

This is a region in which the geostrophic wind can be neglected, but into which free shelf waves are propagating from outside. The integral (4.6) now reduces to

$$gp(y,t) = \sigma D_0 e^{i(\omega t + \frac{1}{2}\pi\delta)} \left[\alpha_0 L + i \sum_{n=1}^{\infty} \frac{\alpha_n}{\kappa_n} e^{i\kappa_n y} (e^{-i\kappa_n L} - 1)\right]. \tag{4.9}$$

Taking the real part, the expression which corresponds to (4.8) in this region is

$$p_1(y,t) = \frac{2\delta\sigma D_0 \alpha_1}{\kappa_1 g} \sin(\tfrac{1}{2}\kappa_1 L) \sin(\omega t + \kappa_1 y - \tfrac{1}{2}\kappa_1 L), \tag{4.10}$$

for the first mode. This is a wave travelling with speed $-\omega/\kappa_1$ which is about 270 cm/sec northwards on the east coast of Australia.

5. Discussion

In view of the remarkable correlation between atmospheric pressure and sea level which was observed by Hamon (1966, figure 2), it is reasonable to conjecture that shelf waves are generated by pressure fluctuations. However, it may be seen that the zero-divergence assumptions leads to an equation (2.5) in which the forcing terms due to pressure are absent. In other words, these forcing terms are found to contain as a factor the small parameter f^2L^2/gh, which, in this context, is about 5×10^{-2}. Thus one would expect that shelf waves forced by pressure variations would have amplitudes which are about two orders of magnitude too small to offer an explanation of the observations. This is not surprising physically, because vorticity is a dominant feature of shelf waves, and one would not expect pressure fluctuations to be a very efficient way of generating vorticity. Actual numerical confirmation of this argument was obtained by one of the authors (Adams 1967), who showed that, for reasonable assumptions regarding pressure fluctuation, the resultant displacement of sea level is not significantly different from that predicted by the isostatic theory.

There have been two attempts to develop a theory in which the shelf is in resonance with atmospheric pressure, thus increasing the magnitudes of the resulting displacements. In his original paper, Robinson (1964) assumes a coastline of infinite length, and a pressure forcing term of the form $\phi = 2\phi_0 \cos\alpha y \cos\omega t$. For a given ω_r there is resonance when $\alpha = \kappa_r$, where $\kappa_r = \omega_r/c$ and c is the speed of shelf waves. The resulting displacements for a frictionless theory are, of course, infinite, and Robinson suggested that a reasonable model including friction, could well explain the observations.

It is desirable at this stage to determine the physical basis of this kind of resonance. The forcing term can be written as

$$\phi = \phi_0 \cos(\alpha y + \omega_r t) + \phi_0 \cos(\alpha y - \omega_r t)$$

and obviously, when $\alpha = \kappa_r$, the first term matches shelf waves in speed and direction. Now suppose a disturbance is generated at $y = 0$ at $t = 0$, then by time $t = 2\pi/\omega_r$ it will have travelled a distance $y = 2\pi/\kappa_r$ and will then reinforce the waves being currently generated. Thus all the waves travelling from $y = -\infty$ will reinforce and infinite amplitudes result. Clearly, this type of resonance is significant only if the waves can travel for a sufficient number of wavelengths within the forcing region so that significant reinforcement can take place. Now it has been shown that the relevant wavelengths are about 2000 km, and the relevant Australian coastlines are hardly longer than this. An equally important limitation is that weather systems have widths which are not much greater than the wavelength so that even in America it is quite impossible to find a coast where pressure and shelf wave can remain in phase long enough to give sufficient reinforcement. Mysak (1967*b*) has generalized Robinson's theory, but an infinite coastline is still needed to give the required resonance.

More recently, Mysak (1967*a*) has proposed a model consisting of a narrow shelf on a circular continent. In this case resonance can be expected at certain frequencies corresponding to the reinforcement of waves after travelling around the circumference of the circle. Whilst this is a mathematically sound theory, it is difficult to see how it can be used to explain the observations for the following reasons. (i) The shelf is not continuous, since both Tasmania and New Guinea offer impenetrable obstacles to the propagation of shelf waves. This is supported by the fact that observations of sea level at Hobart do not depart significantly from the isostatic value, and the regression coefficient at Eden is much nearer the 'expected' value -1, than those observed further north. (ii) At a speed of 350 cm/sec shelf waves would take some weeks to complete the circumference of Australia, and by then they would suffer sufficient decay due to friction to preclude reasonable reinforcement. In actual fact, Mysak assumed a very low friction coefficient to obtain reasonable amplitudes at resonant frequencies. (iii) It is difficult to see how a resonance theory can explain both the existence of the broad peak in the frequency spectrum and the opposite phases of the waves on the east and west coasts.

The question of the generation of shelf waves by wind was raised by Hamon (1966) who showed that the 'set up' due to the geostrophic wind on a shelf of width 50 km and uniform depth 100 m is an order of magnitude too small. However, a sloping shelf is quite a different matter, because for barotropic motion the stress of a wind parallel to the shelf is quite an efficient way of generating the necessary vorticity.

The theoretical results in this paper give considerable qualitative support to the theory that the longshore component of the geostrophic wind accounts for the observed shelf waves. Many of the features which appear in table 1 in Hamon's (1966) paper can be given a reasonable explanation. For instance, the comparatively larger displacements on the west coast are explained by the wider shelf, and the small departure from the isostatic value at Eden is explained by the fact that the waves start only just south of there. Note also that it can be shown that, in (4.8), $p_1(y,t)$ is only a slowly varying function of the frequency. This would explain why little or no difference can be detected in the regression coefficient between summer, when the frequency spectrum has a peak corresponding to a period of nine days, and winter, when the peak corresponds to a 5-day period.

There remains the question of the high correlation between sea level and atmospheric pressure. One can only conclude that at periods in the spectrum of between 5 and 9 days, the wind fluctuations are almost entirely due to the geostrophic and gradient components, which are, of course, very highly correlated with pressure.

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